§2 Integration for bounded measurable functions with $m(E) < +\infty$.

Let $m(E) < +\infty$ and

Th 1. $BIF(E) := \{ f \in BF(E) : \overline{\int_E} f = \underline{\int_E} f = \int_E f \}$.

(i) $BIF(E)$ is a vector space and $f \mapsto \int_E f$ is linear on $BIF(E)$.

(ii) If $f \in BIF(E)$ and $E = E_1 \cup_\circ E_2$ with $E_1, E_2 \in \mathfrak{M}$, $E_1 \cap E_2 = \emptyset$ then $\int_E f = \int_{E_1} f + \int_{E_2} f$

(iii) If $f, g \in BIF(E)$ and $f \leq g$ a.e on E then $\int_E f \leq \int_E g$.

(iv) Each $f \in BIF(E)$ is measurable.

(v) Let $f, g \in BIF(E)$ be such that $f \leq g$ a.e. on E and $\int_E f = \int_E g$ then $f = g$ a.e on E.

Proof (i) Let $f \in BIF(E)$ then $\underline{\int_E} f = \overline{\int_E} f$ and

$$\overline{\int_E}(-f) = -\underline{\int_E} f = -\overline{\int_E} f = \underline{\int_E}(-f)$$ [you should check the 1st equality]

So $-f \in BIF(E)$ and $\int_E (-f) = -\int_E f$. Also, $\forall \alpha > 0$

$\overline{\int_E}(\alpha f) = \alpha \overline{\int_E} f = \alpha \underline{\int_E} f = \underline{\int_E}(\alpha f)$, showing that

$\alpha f \in BIF(E)$ and $\int_E (\alpha f) = \alpha \int_E f$. Finally, let $f, g \in BIF(E)$. Then [you should check the first/last inequality]

$$\overline{\int_E}(f+g) \leq \overline{\int_E} f + \overline{\int_E} g = \underline{\int_E} f + \underline{\int_E} g \leq \underline{\int_E}(f+g)$$

showing that $f+g \in BIF(E)$ and $\int_E (f+g) = \int_E f + \int_E g$.

(ii). See Prop 1^* (ii)

(iii). Let $\psi \in \mathcal{S}(E)$ such that $g \leq \psi$ on E. Then $f \leq \psi$ a.e. on E so $\overline{\int_E} f \leq \psi$. Taking "inf" over all such ψ, we arrive at $\overline{\int_E} f \leq \overline{\int_E} g$ (and $\int_E f \leq \int_E g$ when $f, g \in BIF(E)$).

(iv) Let $\overline{\int_E} f = \underline{\int_E} f$ (as $f \in BIF(E)$), and let $n \in \mathbb{N}$. Then $\exists\ \varphi_n, \psi_n \in \mathcal{S}(E)$ such that $\varphi_n \leq f \leq \psi_n$ on E such that

$$\underline{\int_E} f - \frac{2}{n} < \int_E \varphi_n$$

$$\overline{\int_E} f + \frac{2}{n} > \int_E \psi_n$$

and so $\int_E \psi_n - \int_E \varphi_n < \frac{1}{n}$, hence

$$0 \leq \int_E (\psi_n - \varphi_n) < \frac{1}{n}$$

Let $\bar{\varphi} = \bigvee_{n\in\mathbb{N}} \varphi_n$ (ptwise) and $\bar{\psi} = \bigwedge_{n\in\mathbb{N}} \psi_n$.

Then $\bar{\varphi}, \bar{\psi}$ are measurable (why?) and

$\varphi_n \leq \bar{\varphi} \leq f \leq \bar{\psi} \leq \psi_n$. Shall show that

$\bar{\varphi} = \bar{\psi}$ a.e on E (so $\bar{\varphi} = f = \bar{\psi}$ a.e on E

and f is measurable). Let

$$\Delta := \{x \in E : \bar{\varphi}(x) \neq \bar{\psi}(x)\} = \{x \in E : \bar{\varphi}(x) < \bar{\psi}(x)\}$$

and

$$\Delta_\varepsilon := \{x \in E : \varepsilon \leq \bar{\psi}(x) - \bar{\varphi}(x)\}$$

Then $\Delta, \Delta_\varepsilon$ are measurable ($\varepsilon > 0$) and

$\Delta = \bigcup_{m\in\mathbb{N}} \Delta_{1/m}$ so it suffices to show that

$m(\Delta_\varepsilon) = 0 \ \forall \varepsilon > 0$. Since $\bar{\psi} - \bar{\varphi} \leq \psi_n - \varphi_n$,

one notes that $\varepsilon \leq \psi_n - \varphi_n$ on Δ_ε and so

$$\int_{\Delta_\varepsilon} \varepsilon \leq \int_{\Delta_\varepsilon} (\psi_n - \varphi_n) \leq \int_E (\psi_n - \varphi_n) < 1/n,$$

($0 \leq \psi_n - \varphi_n$ on $E \setminus \Delta_\varepsilon$ is also used)

i.e $0 \leq \varepsilon\, m(\Delta_\varepsilon) < 1/n \ \forall n \in \mathbb{N}$. Hence

$\varepsilon \cdot m(\Delta_\varepsilon) = 0$ and so $m(\Delta_\varepsilon) = 0$ as

required to show.

(v). Let $h = g - f$ $(\in BIF(E))$. Then h is measurable (by (iv)) and $\int_E h = 0$ (by (ii)). and $0 \le h$ a.e on E. We need to show that $h = 0$ a.e on E. Similar as in (iv) we only need to show that $m(\Delta_\varepsilon) = 0$ $(\forall \varepsilon > 0)$ where

$$\Delta_\varepsilon := \{x \in E : \varepsilon \le h(x)\}$$

Since h is measurable, $\Delta_\varepsilon \in \mathfrak{M}$ and, by linearity,

$$0 = \int_E h = \int_{\Delta_\varepsilon} h + \int_{E \setminus \Delta_\varepsilon} h \geq \int_{\Delta_\varepsilon} h + 0 \geq \varepsilon \cdot m(\Delta_\varepsilon)$$

So $m(\Delta_\varepsilon) = 0$.

Th2. Let $m(E) < +\infty$, $f \in BF(E)$. Then

$$f \in BIF(E) \iff f \text{ is measurable.}$$

Pf $\Rightarrow$ already done in the preceding theorem (part (iv)).

$\Leftarrow$ Suppose f is measurable. Since f is also bounded, it follows from Littlewood's 2nd principle that, $\forall \varepsilon > 0$, $\exists$ simple functions $\varphi, \psi \in \mathcal{S}(E)$ such that $\varphi \le f \le \psi$ and $\psi - \varphi < \varepsilon$ on E. Then

$$\overline{\int_E} f \le \int_E \psi \le \int_E (\varphi + \varepsilon) = \int_E \varphi + \varepsilon\, m(E) \le \underline{\int_E} f + \varepsilon\, m(E)$$

Since $\varepsilon > 0$ is arbitrary and $m(E) < +\infty$, we then have

$\overline{\int_E} f \leq \underline{\int_E} f$ so $f \in BIF(E)$.

Th3. (Bounded Convergence Th). please see my typed notes.

For future convenience, below I shall rephrase this theorem ~~and~~ a slight more generally: let $E \in \mathcal{M}$ (so $m(E) \leq +\infty$) and let $\mathcal{B}_0(E)$ denote the set of measurable bounded functions on E vanishing outside a set of finite measure (and vanishing outside E) : $\forall f \in \mathcal{B}_0(E)$, $\exists A \subseteq E$ with $m(A) < +\infty$ such that $f = 0$ on $E \setminus A$ in addition to being bounded and measurable. In this case one defines

$$\int_E f := \int_A f$$

You should be able to check that this is well-defined: if $A' \subseteq E$ is another measurable subset of finite measure s.t. $f = 0$ on $E \setminus A'$ then

$$\int_A f = \int_{A'} f. \left(= \int_{A \cap A'} f \right)$$

Th 3* ("Extended Version" of BC Theorem)
Let (f_n) be a sequence of measurable functions on $E \in \mathcal{M}$, almost everywhere convergent to (measurable function) f on E. Suppose $\exists$ a measurable subset E_0 of E with finite measure such that

$$f_n = 0 \text{ on } E \setminus E_0 \quad (\forall n \in \mathbb{N})$$

and suppose further that $\exists$ real $M > 0$ s.t.

(#) $\quad |f_n| \le M$ on $E \quad (\forall n \in \mathbb{N})$.

Then $\int_E f_n \to \int_E f$

Note. Of course (#) can also be relaxed to hold a.e. on E (instead of everywhere on E).